# Attitude Control of Quad-Rotor UAVs Using An Intuitive Kinematics Model

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***Abstract*—This paper presents the work on attitude control of quad-rotor UAVs applying an intuitive kinematics representation, called rotation vector. There are three elements in the rotation vector which has clear physical meaning of the rotations and avoids the singularity problem of Euler angles and the unity norm constraint problem of quaternions. Basic definition of the rotation vector and its relation with the object body angle velocity is introduced and used in the 6DOF quad-rotor dynamics. Based on the property that the rotation vector rate is equivalent to the body angle velocity when the rotation is small, a simple and intuitive attitude reference is proposed. A proportional-derivative (PD) law is used by integrating the new attitude reference for the attitude control of quad-rotor UAVs. Simulation results prove the efficiency of the new method which provides a new model with intuitive physical meaning for quad-rotor UAVs.**

## I. Introduction

Unmanned aerial vehicles (UAVs) have been prevalently employed in numerous defense- and civil-related applications over the last two to three decades. Among various types of UAVs, miniature quad-rotor UAV (an example, constructed at the Robotics Institute at Khalifa University, is shown in Fig. 1) has gained particular popularity due to its unique features such as compact size, good agility, high maneuverability and vertical take-off and landing (VTOL). For now they are widely studied and used in surveillance, first responder for fire and civil accidents, inspection, photography and mapping [1].

Quad-rotor UAVs generally have a cross shape with four rotors arranged at the four ends of the cross. By controlling speeds of the four rotors in variable combinations, under controlled strategy is applied to control four degrees of freedom (DOFs) in realizing the 6-DOF translations and orientations. Quad-rotor UAVs are single rigid bodies when considering their dynamics which provides a good platform for applying various control laws from linear methods including LQR [2, 3], LQG [4], PD [5] and PID [6], to non-linear ones using hierarchical controllers with Lyapunov method [7-9], sliding-mode technique [10, 11], and predictive strategy [12, 13].

However, most of the above work used Euler angle model which is one of the two most popular orientation representation models including Euler angles and quaternions. The main disadvantages of Euler angles are [14] that they have singularities in angular velocity calculation and that they are less accurate than unit quaternions when used to integrate incremental changes in attitude over time. For unit quaternions [14], their four parameters do not have intuitive physical meanings and they must have unity norm to represent a pure rotation. The quadratic unity norm constraint is particularly problematic in optimization. The rotation vector used in this paper lacks both the singularities of the Euler angles and the quadratic constraint of the unit quaternion. In addition, it also shows intuitive physical meaning in the attitude representation of quad-rotor UAVs, which results in a new attitude reference in the quad-rotor UAV control.

Figure 1. Khalifa University UAV (KUAV)

The following of this paper is organized as: the rotation vector and its corresponding rotation matrix are introduced while the kinematics of quad-rotor UAVs is developed in section II. Based on these, section III presents the dynamics model of quad-rotor UAVs. Then a new attitude reference is proposed in section IV and a PD controller is illustrated. To verify the theoretical analysis, section V shows the simulation results and conclusions make up section VI.

## II. KINEMATICS MODELING

### A. Coordinate systems of quad-rotor UAVs

Generally, an inertia coordinate system *oxyz* is set on the ground to be the reference while a body coordinate system *o′x′y′z′* is attached on the UAV as shown in Fig. 2. The four rotor thrust $F_i$ ($i$=1,2,3,4) are constantly parallel to the $z'$-axis on the body. The difference of the two pairs, ($F_1$, $F_3$) and ($F_2$, $F_4$), leads to variable attitudes and translations of the UAV which can be represented by the rotation vector as below.

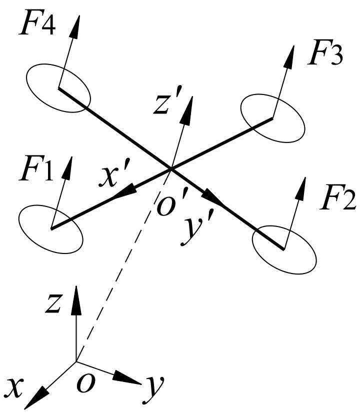


Figure 2. Quad-rotor UAV coordinate systems

### B. The rotation vector and quad-rotor UAV kinematics

The rotation vector is defined as a function of the rotation axis and angle of a rotation,

$$\boldsymbol{k} = \theta * \boldsymbol{n} = (k_x, k_y, k_z) \tag{1}$$

Thus, the magnitude of $\boldsymbol{k}$ is the rotation angle $\theta$ and the unit vector of $\boldsymbol{k}$ is the rotation axis $\boldsymbol{n}$. Based on this, the rotation between the body coordinate system and the inertial coordinate system is given as:

$$\mathbf{R} = \frac{1}{\theta^2}\begin{bmatrix} (k_x^2 - k_y^2 - k_z^2)\mathrm{s}_{\theta/2}^2 + \theta^2\mathrm{c}_{\theta/2}^2 & 2\mathrm{s}_{\theta/2}(k_xk_y\mathrm{s}_{\theta/2} + \theta k_z\mathrm{c}_{\theta/2}) & 2\mathrm{s}_{\theta/2}(k_xk_z\mathrm{s}_{\theta/2} - \theta k_y\mathrm{c}_{\theta/2}) \\ 2\mathrm{s}_{\theta/2}(k_xk_y\mathrm{s}_{\theta/2} - \theta k_z\mathrm{c}_{\theta/2}) & (k_y^2 - k_x^2 - k_z^2)\mathrm{s}_{\theta/2}^2 + \theta^2\mathrm{c}_{\theta/2}^2 & 2\mathrm{s}_{\theta/2}(k_yk_z\mathrm{s}_{\theta/2} + \theta k_x\mathrm{c}_{\theta/2}) \\ 2\mathrm{s}_{\theta/2}(k_xk_z\mathrm{s}_{\theta/2} + \theta k_y\mathrm{c}_{\theta/2}) & 2\mathrm{s}_{\theta/2}(k_yk_z\mathrm{s}_{\theta/2} - \theta k_x\mathrm{c}_{\theta/2}) & (k_z^2 - k_y^2 - k_x^2)\mathrm{s}_{\theta/2}^2 + \theta^2\mathrm{c}_{\theta/2}^2 \end{bmatrix} \tag{2}$$

where $\mathrm{s}_{\theta/2}$, $\mathrm{c}_{\theta/2}$, $\mathrm{s}_{\theta/2}^2$ and $\mathrm{c}_{\theta/2}^2$ represent sin($\theta$/2), cos($\theta$/2) and their squares.

Then, the angular velocity $\boldsymbol{\omega}$ of the UAV can be related to the rotation vector rate $\dot{\boldsymbol{k}}$ in the inertial coordinate system by:

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 2\,\mathbf{W}\begin{bmatrix} \dot{k}_x \\ \dot{k}_y \\ \dot{k}_z \end{bmatrix} = 2\,\mathbf{W}\,\dot{\boldsymbol{k}} \tag{3}$$

where

$$\mathbf{W} = \frac{1}{2\theta^4}\begin{bmatrix} \theta^2k_x^2 - \theta k_x^2\mathrm{s}_\theta + \theta^3\mathrm{s}_\theta & \theta^2k_xk_y - \theta k_xk_y\mathrm{s}_\theta - 2\theta^2k_z\mathrm{s}_{\theta/2}^2 & \theta^2k_xk_z - \theta k_xk_z\mathrm{s}_\theta + 2\theta^2k_y\mathrm{s}_{\theta/2}^2 \\ \theta^2k_xk_y - \theta k_xk_y\mathrm{s}_\theta + 2\theta^2k_z\mathrm{s}_{\theta/2}^2 & \theta^2k_y^2 - \theta k_y^2\mathrm{s}_\theta + \theta^3\mathrm{s}_\theta & \theta^2k_yk_z - \theta k_yk_z\mathrm{s}_\theta - 2\theta^2k_x\mathrm{s}_{\theta/2}^2 \\ \theta^2k_xk_z - \theta k_xk_z\mathrm{s}_\theta - 2\theta^2k_y\mathrm{s}_{\theta/2}^2 & \theta^2k_yk_z - \theta k_yk_z\mathrm{s}_\theta + 2\theta^2k_x\mathrm{s}_{\theta/2}^2 & \theta^2k_z^2 - \theta k_z^2\mathrm{s}_\theta + \theta^3\mathrm{s}_\theta \end{bmatrix},$$

$\mathrm{s}_\theta$ denotes sin($\theta$).

For the UAV translation velocity, there is

$$\dot{\boldsymbol{p}} = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \mathbf{R}\,\boldsymbol{v} = \mathbf{R}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{4}$$

where $\boldsymbol{p}$ is the UAV position vector in the inertia coordinate system, $\boldsymbol{v}$ is the velocity vector in the body coordinate system.

Thus, the translation acceleration follows the derivative relation of (4) as:

$$\ddot{\boldsymbol{p}} = \mathbf{R}\,\dot{\boldsymbol{v}} + \dot{\mathbf{R}}\,\boldsymbol{v} = \mathbf{R}\,\dot{\boldsymbol{v}} + \mathbf{R}\,S(\boldsymbol{\omega})\boldsymbol{v} \tag{5}$$

where $S(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$.

## III. QUAD-ROTOR DYNAMICS MODELING

The 6-DOF rigid-body dynamics of the UAV is expressed in the body coordinate system by the following Newton-Euler equations:

$$m\dot{\boldsymbol{v}} + mS(\boldsymbol{\omega})\boldsymbol{v} - m\mathbf{R}^T\boldsymbol{g} = \boldsymbol{F} = \boldsymbol{F}_{rotor} - \boldsymbol{F}_{aero} \tag{6}$$

and

$$\mathbf{M}\dot{\boldsymbol{\omega}} + S(\boldsymbol{\omega})(\mathbf{M}\boldsymbol{\omega}) = \boldsymbol{T} = \boldsymbol{T}_{rotor} - \boldsymbol{T}_{aero} \tag{7}$$

where $m$ is the UAV mass and $\mathbf{M}$ is its inertial matrix, $\boldsymbol{F}$ and $\boldsymbol{T}$ are the sum of external forces and torques experienced by the UAV at its center of mass. $\boldsymbol{F}_{\mathrm{rotor}}$ is the sum force from the four rotors of the UAV, $\boldsymbol{F}_{\mathrm{aero}}$ is the aerodynamic friction force, $\boldsymbol{g}$ = [0,0,-9.81m/s$^2$]$^T$ is the gravity acceleration, $\boldsymbol{T}_{\text{rotor}}$ is the torque from the rotors and the $\boldsymbol{T}_{\text{aero}}$ is the aerodynamic friction torque. They have the following expressions:

$$\boldsymbol{F}_{rotor} = \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{4} F_i \end{bmatrix}, \quad \boldsymbol{F}_{aero} = \mathbf{K}_F\, \boldsymbol{v},$$
$$\boldsymbol{T}_{rotor} = \begin{bmatrix} d(F_2 - F_4) \\ d(F_3 - F_1) \\ \lambda(F_2 - F_1 + F_4 - F_3) \end{bmatrix}, \quad \boldsymbol{T}_{aero} = \mathbf{K}_T\, \boldsymbol{v}, \tag{8}$$

where $F_i$ ($i$=1,2,3,4) is the thrust of rotor $i$ as shown in Fig. 2, $\mathbf{K}_F$ and $\mathbf{K}_T$ are diagonal matrix aerodynamic coefficients.

Substituting the kinematics model in section 2 to (6), the system dynamics model can be summarized as following:

$$\begin{cases} \ddot{\boldsymbol{p}} = \mathbf{R}\,\boldsymbol{F} / m + \boldsymbol{g} \\ \boldsymbol{\omega} = 2\,\mathbf{W}\,\dot{\boldsymbol{k}} \\ \dot{\boldsymbol{\omega}} = \mathbf{M}^{-1}(\boldsymbol{T} - S(\boldsymbol{\omega})(\mathbf{M}\,\boldsymbol{\omega})) \end{cases} \tag{9}$$

## IV. Attitude Control Algorithm

### A. New Attitude Reference

For a very small $\theta$, from (3) there is:

$$\lim_{\theta\to 0} \mathbf{W} = \lim_{\theta\to 0} \frac{1}{4} \begin{bmatrix} 2 & -k_z & k_y \\ k_z & 2 & -k_x \\ -k_y & k_x & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

Then,

$$\lim_{\theta\to 0} \boldsymbol{\omega} = \lim_{\theta\to 0} 2\,\mathbf{W}\,\dot{\boldsymbol{k}} = \dot{\boldsymbol{k}} \tag{11}$$

Thus, when the rotation angle is very small the rotation vector rate can be taken equivalent to the angle velocity. If the rotation is about a constant direction, the two are the same as

$$\boldsymbol{\omega} = 2\,\mathbf{W}\,\dot{\boldsymbol{k}} = 2\,\mathbf{W}\,\boldsymbol{n}\dot{\theta} = \boldsymbol{n}\dot{\theta} \tag{12}$$

Similarly when $\theta$ is close to zero, from (2) there is:

$$\lim_{\theta\to 0} \mathbf{R} = \begin{bmatrix} 1 & k_z & -k_y \\ -k_z & 1 & k_x \\ k_y & -k_x & 1 \end{bmatrix} \tag{13}$$

For two close coordinate systems ${}_I^A\mathbf{R}$ (the actual body coordinate system with respect to the inertia coordinate system) and ${}_I^R\mathbf{R}$ (the reference body coordinate system with respect to the inertia coordinate system), the relative rotation can be expressed as:

$${}_R^A\mathbf{R} = \begin{bmatrix} 1 & \Delta{}_R^A k_z & -\Delta{}_R^A k_y \\ -\Delta{}_R^A k_z & 1 & \Delta{}_R^A k_x \\ \Delta{}_R^A k_y & -\Delta{}_R^A k_x & 1 \end{bmatrix} \tag{14}$$

which is rotation matrix of the actual coordinate system with respect to the reference coordinate system with $\Delta{}_R^A\boldsymbol{k} = (\Delta{}_R^A k_x, \Delta{}_R^A k_y, \Delta{}_R^A k_z)$.

Thus the rotation vector $\Delta{}_R^A\boldsymbol{k}$ has a linear relation with the rotation matrix ${}_R^A\mathbf{R}$ following the fact that an infinitesimal rotation of one coordinate system with respect to another one is additive and commutative [15]. Since the two coordinate system are close, there is

$$\Delta{}_R^A\boldsymbol{k} = {}_I^A\boldsymbol{k} - {}_I^R\boldsymbol{k} \tag{15}$$

This gives a natural representation of orientation difference which provides the direction for the control to push the actual attitude to the desired one.

### B. Attitude Control Law

Based on (15), a proportional-derivative (PD) controller can be planned by taking the rotation vector difference as the proportion part and the angular velocity difference for the derivative part as:

$$\boldsymbol{T}_{im} = \mathbf{K}_{Tp}({}_I^A\boldsymbol{k} - {}_I^R\boldsymbol{k}) + \mathbf{K}_{Td}({}_I^A\boldsymbol{\omega} - {}_I^R\boldsymbol{\omega}) \tag{16}$$

where $\mathbf{K}_{Tp}$ and $\mathbf{K}_{Td}$ are the orientation compliance and damping coefficient matrices.

## V. Simulation

In this section, some simulation results will be illustrated to verify the effectiveness of the proposed attitude reference and control strategy. The physical parameters of the quad-rotor UAV is set as: $m$ = 2$kg$, $I_x$=$I_y$=$I_z$/2 = 1.2416 $Nm.s^2/rad$, $d$=0.2$m$. The desired attitude trajectory of the quadarotor UAV is a rotation about an axis $\boldsymbol{n}$=(0.433, 0.75, 0.5) while the rotation angle follows the periodic function as:

$$\begin{cases} \gamma = (4/2\pi)^2 \sin(2\pi t/4) + 0.1 \\ \dot{\gamma} = (4/2\pi)\cos(2\pi t/4) \\ \ddot{\gamma} = -\sin(2\pi t/4) \end{cases} \tag{17}$$

The UAV starts from initial orientation with $\boldsymbol{k}$=**0** and **R**=**I** of which the body coordinate is in line with the inertia coordinate system. The simulation results are listed in Fig. 3 with the PD gain as $\mathbf{K}_{TP}$=diagno[90, 90, 90], $\mathbf{K}_{Td}$ =diagno[20, 20, 20]. It can be seen that with the initial orientation difference 0.1 rad between the actual and the desired rotation from (17), the rotation vector elements ($k_x$, $k_y$, $k_z$) can follow the desired values exactly after 1.8s while the angular velocity elements in all the three directions ($w_x$, $w_y$, $w_z$) converge in about 0.8s to follow the desired trajectory. In the whole process, the input torques for the three directions reduce from around 14Nm to stabilize at around 0.15Nm. The simulation proves the efficiency of the proposed attitude control method.

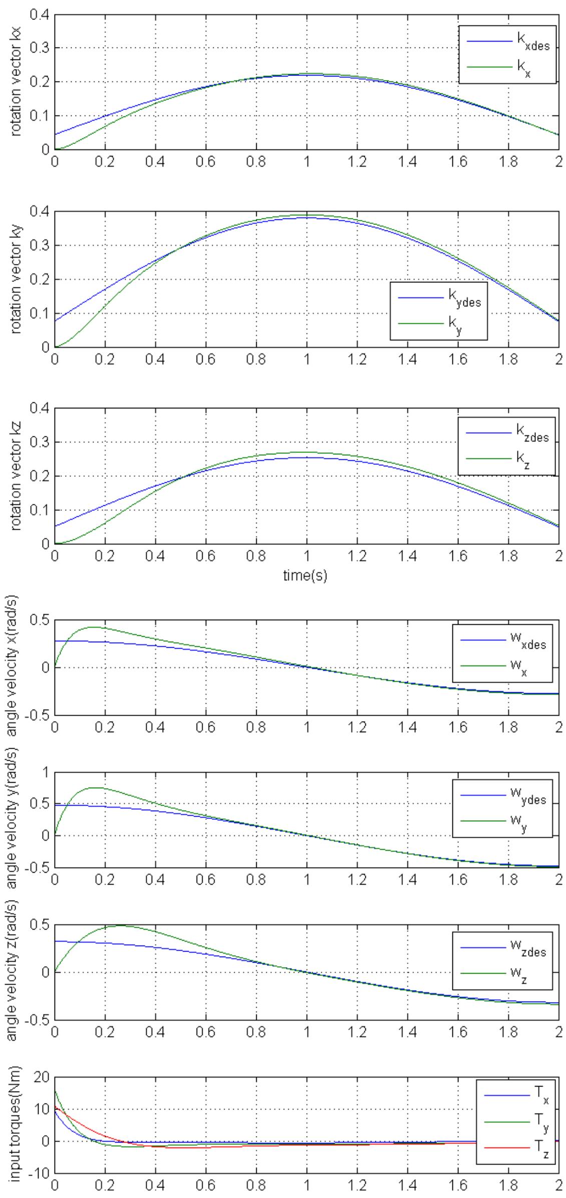


Figure 3. Simulation results

## VI. Conclusions

This paper introduced the rotation vector in quad-rotor attitude control. By investigating the kinematics property, it was found that when the rotation is relatively small, the change rate of the attitude representation is close to the body angular velocity. This was used to be a reference for attitude control between the actual and the desired motion. The new method avoided the singularity problem of Euler angle model by applying the body angular velocity directly which could also improve the processing speed considering the calculation in real time. A proportional-derivative (PD) control law was used in the simulation which verified the effectiveness of the new modeling and control method. Future work will focus on the experiment implementation with the KUAV platform built at the Robotics Institute at Khalifa University as shown in Fig. 1.

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